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Young diagrams and N-soliton solutions of the KP equation

Yuji Kodama

Department of Mathematics, Ohio State University, Columbus, OH 43210, USA

E-mail: kodama@math.ohio-state.edu

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Abstract
We consider N-soliton solutions of the KP equation,

$$(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0.$$

An N-soliton solution is a solution $u(x, y, t)$ which has the same set of N line soliton solutions in both asymptotics $y \to \infty$ and $y \to -\infty$. The N-soliton solutions include all possible resonant interactions among those line solitons. We then classify those N-soliton solutions by defining a pair of N numbers $(\mathbf{n}^+, \mathbf{n}^-)$ with $\mathbf{n}^\pm = \left(n_1^\pm, \ldots, n_N^\pm\right), n_j^\pm \in \{1, \ldots, 2N\}$, which labels N line solitons in the solution. The classification is related to the Schubert decomposition of the Grassmann manifolds $\mathrm{Gr}(N, 2N)$, where the solution of the KP equation is defined as a torus orbit. Then the interaction pattern of N-soliton solution can be described by the pair of Young diagrams associated with $(\mathbf{n}^+, \mathbf{n}^-)$. We also show that N-soliton solutions of the KdV equation obtained by the constraint $\partial u/\partial y = 0$ cannot have resonant interaction.

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1. Introduction

In this paper, we consider a family of exact solutions of the KP equation,

$$\frac{\partial}{\partial x}\left(-4\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x}\right) + 3\frac{\partial^2 u}{\partial y^2} = 0.$$

The solution $u(x, y, t)$ is obtained from the τ-function $\tau(x, y, t)$ as

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2}\log\tau(x, y, t).$$

It is well known that some solutions can be obtained by the Wronskian form (for example, see [2] or the appendix in [1]),

$$\tau = \mathrm{Wr}(f_1, \ldots, f_N) := \begin{vmatrix} f_1^{(0)} & \cdots & f_N^{(0)} \\ \vdots & \ddots & \vdots \\ f_1^{(N-1)} & \cdots & f_N^{(N-1)} \end{vmatrix}, \tag{1.1}$$

where $f_i^{(n)} = \partial^n f_i/\partial x^n$, and $\{f_i(x, y, t) \mid i = 1, \ldots, N\}$ is a linearly independent set of N solutions of the equations,

$$\frac{\partial f_i}{\partial y} = \frac{\partial^2 f_i}{\partial x^2} \qquad \frac{\partial f_i}{\partial t} = \frac{\partial^3 f_i}{\partial x^3}.$$

Throughout this paper, we assume $f_i(x, y, t)$ to have the form

$$f_i = \sum_{j=1}^{M} a_{ij}\, \mathrm{e}^{\theta_j}, \qquad \text{for} \quad i = 1, \ldots, N, \quad \text{and} \quad M > N, \tag{1.2}$$

with some constants a_{ij} which define the $N \times M$ matrix $A_{(N,M)} := (a_{ij})$, and the phase functions θ_j are given by

$$\theta_j(x, y, t) = -k_j x + k_j^2 y - k_j^3 t + \theta_j^0 \qquad \text{for} \quad j = 1, \ldots, M. \tag{1.3}$$

Here k_j and θ_j^0 are arbitrary constants, and throughout this paper we assume k_j being ordered as

$$k_1 < k_2 < \cdots < k_M.$$

Then choosing particular forms for the matrix $A_{(N,M)}$, one can obtain several exact solutions of the KP equation. As the simplest case with $N = 1$ and $M = 2$, i.e. $\tau = f_1 = a_{11}\,\mathrm{e}^{\theta_1} + a_{12}\,\mathrm{e}^{\theta_2}$ with $a_{11}a_{12} > 0$, we have 1-soliton solution,

$$u = 2\frac{\partial^2}{\partial x^2}\log\tau = \frac{1}{2}(k_1 - k_2)^2\,\mathrm{sech}^2\,\frac{1}{2}(\theta_1 - \theta_2). \tag{1.4}$$

Here we assumed $a_{11}, a_{12} > 0$ and absorbed them into the constants θ_j^0, so that we have $a_{11} = a_{12} = 1$. Note here that if $a_{11}a_{12} < 0$, then the τ-function has zeros and the solution blows up at some points in the x–y plane. In the x–y plane, 1-soliton solution describes a plane wave $u = \Phi(k_x x + k_y y - \omega t)$ having the wavenumber vector $\mathbf{k} = (k_x, k_y)$ and the frequency ω,

$$\mathbf{k} = \left(-k_1 + k_2, k_1^2 - k_2^2\right) \qquad \omega = k_1^3 - k_2^3.$$

Here $(\mathbf{k}, \omega)$ satisfies the dispersion relation, $4\omega k_x + k_x^4 + 3k_y^2 = 0$. We refer to the 1-soliton solution as a line soliton, which is expressed by a line $\theta_1 = \theta_2$ in the x–y plane for a fixed t. Then the slope of the line $c = \mathrm{d}x/\mathrm{d}y = -k_y/k_x$ represents the velocity of the line soliton in the x-direction with respect to y; that is, $c = 0$ indicates the direction of the positive y-axis. As in paper [1], we also call a line soliton $[i, j]$-soliton, if the soliton is parametrized by the pair (k_i, k_j). In the case of (1.4), the soliton is $[1, 2]$-soliton with the velocity $c = k_1 + k_2$. Then the ordinary N-soliton solution consisting of N line solitons, $[2j - 1, 2j]$-soliton with $j = 1, \ldots, N$, is obtained from the following matrix $A_{(N,M)}$ with $M = 2N$ [5],

$$A_{(N,2N)} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}.$$

Note here that all nonzero entries are normalized to be one by choosing appropriate constants θ_j^0, and any $N \times N$ minor of the matrix $A_{(N,2N)}$ is non-negative. The fact that all $N \times N$ minors are non-negative (or non-positive) is sufficient for the solution to be non-singular (see below).

Also in paper [1], we found that if all the minors are nonzero and have the same sign, the solution u gives an $(M-N, N)$-soliton solution consisting of $M-N$ incoming line solitons as $y \to -\infty$ and N outgoing line solitons as $y \to \infty$. The set of τ-functions with different size N then provide the solution of the Toda lattice. In particular, if $M = 2N$, we have N-soliton solution in the sense that the solution has the same set of N line solitons in both asymptotics for $y \to \pm\infty$ (i.e. the sets of incoming and outgoing solitons are the same). However as mentioned in [1], this N-soliton solution is different from the ordinary N-soliton solution, and the interaction of any pair of line solitons is in resonance, i.e. every interaction point of line solitons forms a Y-shape vertex satisfying the resonant condition among those three solitons. In this case, the matrix $A_{(N,2N)}$ can be written in the following row reduced echelon form (RREF),

$$A_{(N,2N)} = \begin{pmatrix} 1 & 0 & \cdots & 0 & * & * & \cdots & * \\ 0 & 1 & \cdots & 0 & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & * & * & \cdots & * \end{pmatrix}$$

where the entries marked by '$*$' are chosen so that all the $N \times N$ minors are nonzero and positive (i.e. all the $*$ of the last row should be all positive, and those in the second row from the bottom are negative and so on). Note here that those $*$ are all nonzero. Then the line solitons in N-soliton solution of the Toda lattice hierarchy are given by $[j, N+j]$-soliton for $j = 1, \ldots, N$.

We then expect that the general case of N-soliton solutions has a mixed pattern consisting of resonant and non-resonant interactions among those line solitons. In this paper, we classify all the possible N-soliton solutions obtained by the τ-function (1.1) with f_j in (1.2) and $M = 2N$. The classification is to determine all possible patterns of interactions, and it is done by constructing a specific form of the coefficient matrix $A_N := A_{(N,2N)}$ in (1.2) for each class of N-soliton solutions. It turns out that a complete classification of N-soliton solutions can be obtained by the complementary pair of ordered sets of N numbers, $\mathbf{n}^+ = \left(n_1^+, \ldots, n_N^+\right)$ and $\mathbf{n}^- = (n_1^-, \ldots, n_N^-)$, which are related to the dominant exponentials in the τ-function for $x \to \pm\infty$ and each soliton is given by $\left[n_j^+, n_j^-\right]$-soliton. Note that with the set $\{1, \ldots, 2N\}$, there are $(2N-1)!!$ number of different sets of N pairs. We then claim that each set of N pairs corresponds to an N-soliton solution, and provides an explicit way to construct all those N-soliton solutions.

It is also well known that a solution of the KP hierarchy is a GL(∞)-orbit on an infinite-dimensional Grassmannian (Sato Grassmannian) [9]. Then the N-soliton solution can be identified as the $2N$-dimensional torus orbit on the Grassmannian Gr$(N, 2N)$, and the coefficient matrix A_N represents a point on Gr$(N, 2N)$. Since the Grassmannian has the Schubert cell decomposition, one can first classify the matrix A_N using the decomposition, i.e. identify the cell A_N belongs to. This classification corresponds to the set $\mathbf{n}^+$, and the $\mathbf{n}^-$ gives a further decomposition of the cells to classify the interaction patterns of N-soliton solutions. We then define a pair of Young diagrams (Y^+, Y^-) associated with the pair of the number sets $(\mathbf{n}^+, \mathbf{n}^-)$. In particular, the number of resonant vertices in the N-soliton solution parametrized by (Y^+, Y^-) is given by

$$\frac{N(N-1)}{2} - |Y^+| - |Y^-|,$$

where $|Y^{\pm}|$ denote the size (degree) of the diagrams, i.e. the number of boxes in the diagrams (theorem 4.3).

The paper is organized as follows. In section 2, we give a general basic structure of the τ-function, and present all types of 2-soliton solutions, which provide the building blocks for the general case of N-soliton solutions. Here we also introduce the pair of numbers $(\mathbf{n}^+, \mathbf{n}^-)$, and the Young diagrams (Y^+, Y^-). In section 3, we briefly introduce the Grassmannian $\mathrm{Gr}(N, M)$ and the Schubert decomposition of $\mathrm{Gr}(N, M)$. We also identify N-soliton solution as a $2N$-dimensional torus orbit on $\mathrm{Gr}(N, 2N)$, and briefly mention that $\mathrm{Gr}(N, 2N)$ contains all possible (m, n)-soliton solutions for $1 \leqslant m \leqslant N$ and $1 \leqslant n \leqslant N$, which are distinguished by the Schubert decomposition. Here (m, n)-soliton is the solution consisting of m incoming solitons for $y \to -\infty$ and n outgoing solitons for $y \to \infty$ (see [1]). In section 4, we describe the structure of the coefficient matrix A_N for each N-soliton solution by prescribing an explicit construction of the matrix A_N. We here define the *N-soliton condition* on the matrix A_N which determines local structures based on the types of 2-soliton interaction in the N-soliton solution. Finally, in section 5, we discuss N-soliton solutions of the KdV equation, and show that the KdV N-soliton solution cannot have resonant interactions.

2. Basic structure of the τ-function and 2-soliton solutions

Let us start with the following lemma which shows the basic structure of the τ-function:

Lemma 2.1. *The τ-function (1.1) with f_j given in (1.2) can be expanded as a sum of exponential functions,*

$$\tau = \sum_{1\leqslant i_1<\cdots<i_N\leqslant M} \xi(i_1, \ldots, i_N) \prod_{1\leqslant j<l\leqslant N} \left(k_{i_j} - k_{i_l}\right) \exp\left(\sum_{j=1}^{N} \theta_{i_j}\right),$$

where $\xi(i_1, \ldots, i_N)$ is the $N \times N$ minor given by the i_jth columns with $j = 1, \ldots, N$ in the matrix $A_{(N,M)} = (a_{ij})$ of (1.2),

$$\xi(i_1, \ldots, i_N) := \begin{vmatrix} a_{1,i_1} & \cdots & a_{1,i_N} \\ \vdots & \ddots & \vdots \\ a_{N,i_1} & \cdots & a_{N,i_N} \end{vmatrix}.$$

Proof. Apply the Binet–Cauchy theorem for the expression (see, for example, [3]),

$$\tau = \left| \begin{pmatrix} E_1^{(0)} & E_2^{(0)} & \cdots & E_M^{(0)} \\ E_1^{(1)} & E_2^{(1)} & \cdots & E_M^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ E_1^{(N-1)} & E_2^{(N-1)} & \cdots & E_M^{(N-1)} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{N1} \\ a_{12} & a_{22} & \cdots & a_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{1M} & a_{2M} & \cdots & a_{NM} \end{pmatrix} \right|,$$

where $E_j^{(n)} = (-k_j)^n E_j$ with $E_j = \mathrm{e}^{\theta_j}$. □

From this lemma, it is clear that if all the minors $\xi(i_1, \ldots, i_N)$ are non-negative (or non-positive), then the τ-function is sign definite and the corresponding solution u is non-singular (recall also the order $k_1 < \cdots < k_M$). This is the first restriction on the matrix $A_{(N,M)}$ for N-soliton solution, and we later impose that all the $N \times N$ minors are non-negative after making $A_{(N,M)}$ to be in the row reduced echelon form.

We now consider the case of N-soliton solutions, that is, we take $M = 2N$. With the ordering of the numbers k_j, i.e. $k_1 < k_2 < \cdots < k_{2N}$, one can find the asymptotic behaviour of the τ-function for $x \to \pm\infty$: for this purpose, we first define the function w given by

$$w(x, y, t) = -\frac{\partial}{\partial x} \log \tau(x, y, t).$$

Suppose that w has the asymptotic form with some numbers $\left\{n_1^{\pm}, \ldots, n_N^{\pm}\right\} \subset \{1, \ldots, 2N\}$,

$$w \longrightarrow \begin{cases} \sum_{j=1}^{N} k_{n_j^+} & \text{as} \quad x \to \infty, \\ \sum_{j=1}^{N} k_{n_j^-} & \text{as} \quad x \to -\infty. \end{cases} \tag{2.1}$$

It is clear that $n_i^+ \neq n_j^+$ and $n_i^- \neq n_j^-$ if $i \neq j$. Then with the sets of N numbers $n_j^{\pm}$ in the asymptotics (2.1), we define the following two N-vectors with the entries from the set $\{1, 2, \ldots, 2N\}$:

Definition 2.1. *A pair* $(\mathbf{n}^+, \mathbf{n}^-)$ *of N-vectors are defined by*

$$\begin{cases} \mathbf{n}^+ = \left(n_1^+, n_2^+, \ldots, n_N^+\right) & \text{with} \quad 1 = n_1^+ < n_2^+ < \cdots < n_N^+ < 2N, \\ \mathbf{n}^- = \left(n_1^-, n_2^-, \ldots, n_N^-\right) & \text{with} \quad n_j^- > n_j^+. \end{cases}$$

The ordering in the set $\mathbf{n}^+$ *is just a convenience, and once we make the ordering in* $\mathbf{n}^+$*, we do not assume a further ordering for the other set* $\mathbf{n}^-$*. We assume that those sets are complementary in the set* $\{1, 2, \ldots, 2N\}$*. As will be explained below, this assumption is necessary to have an N-soliton solution, that is, the solution has the same sets of N line solitons in both asymptotics* $y \to \pm\infty$*.*

With those number sets $\mathbf{n}^+$ and $\mathbf{n}^-$, one can define the pair of Young diagrams (Y^+, Y^-):

Definition 2.2. *We define a pair* (Y^+, Y^-) *associated with the pair* $(\mathbf{n}^+, \mathbf{n}^-)$*:*

- *The* Y^+ *is a Young diagram represented by* $\left(\ell_1^+, \ldots, \ell_N^+\right)$ *with* $\ell_j^+ = n_j^+ - j$ *where each* ℓ_j^+ *represents the number of boxes in a row counting from the bottom (note* $\ell_j^+ \leqslant \ell_{j+1}^+$*).*
- *The* Y^- *is a Young diagram associated with* $(\ell_1^-, \ldots, \ell_N^-)$*, where* ℓ_j^- *is defined as the number of reverses in the sequence* $(n_1^-, \ldots, n_N^-)$ *for each* n_j^-*, i.e.*

$$\ell_j^- = |\{n_k^- \mid n_k^- < n_j^- \text{ for } k > j\}|.$$

Here $|\{S\}|$ *implies the number of elements in the set* $\{S\}$*. Then* ℓ_j^- *represents the number of boxes in a row arranged in the increasing order from the bottom.*

Note that the Young diagram Y^+ is uniquely determined from $\mathbf{n}^+$, but the diagram Y^- is not unique (note that $(\ell_1^-, \ldots, \ell_N^-)$ is unique of course). However, the number of boxes in Y^- gives the number of intersection vertices of N-soliton solution having a particular type of interaction (theorem 4.3).

As a main goal of this paper, we will show in section 4 that for each pair $(\mathbf{n}^+, \mathbf{n}^-)$ one can construct an N-soliton solution consisting of N line solitons, each of which is given by $\left[n_j^+, n_j^-\right]$-soliton for $j = 1, \ldots, N$. The total number of different N-soliton solutions is given by $(2N-1)!!$ which is the number of different sets of pairs $\{(\mathbf{n}^+, \mathbf{n}^-)\}$. Namely, we construct an N-soliton solution which is labelled by $(\mathbf{n}^+, \mathbf{n}^-)$ as follows:

(1) Consider a set of $2N$ numbers $\{1, \ldots, 2N\}$, which are associated with the parameters $k_1 < \cdots < k_{2N}$.
(2) Choose N different pairs from $\{1, \ldots, 2N\}$, and label each pair as $[n_j^+, n_j^-]$ so that the numbers $n_j^\pm$ satisfy
   (i) $n_j^+ < n_j^-$ for $j = 1, \ldots, N$;
   (ii) $1 = n_1^+ < n_2^+ < \cdots < n_N^+$.
(3) Define the pair $(\mathbf{n}^+, \mathbf{n}^-)$ with $\mathbf{n}^\pm = (n_1^\pm, \ldots, n_N^\pm)$, and construct the corresponding matrix $A_N = A_{(N,2N)}$.

Note here that the labelling is unique with a given choice of pairs, and the corresponding N-soliton solution consists of $[n_j^+, n_j^-]$-solitons for $j = 1, \ldots, N$. Item (3) will be given in section 4.

In order to classify those N-soliton solutions, we first note

Lemma 2.2. *For each given ordered set* $\mathbf{n}^+$*, the number of choices of* $\mathbf{n}^-$ *is given by*

$$m_{\mathbf{n}^+} = \prod_{j=1}^{N} (2j - n_j^+).$$

Proof. Since $n_N^- > n_N^+$, there are $(2N - n_N^+)$ numbers of possible choices of n_N^-. Having made a choice of n_N^-, one has $2(N-1) - n_{N-1}^+$ many choices for n_{N-1}^-. Now repeating this, the result is obvious. □

This lemma provides the number of different N-soliton solutions having the same $\mathbf{n}^+$, that is, the functions $w = -(\partial/\partial x)\log\tau$ for those solutions have the same asymptotic values for $x \to \pm\infty$. As a corollary of this lemma, we also have the following identity:

$$(2N-1)!! = \sum_{\mathbf{n}^+} m_{\mathbf{n}^+}.$$

We also have the following lemma for the ordered set $\mathbf{n}^+$:

Lemma 2.3. *Each number* n_j^+ *in the* $\mathbf{n}^+$ *is limited as*

$$n_j^+ \leqslant 2j - 1 \qquad \text{for} \quad j = 1, \ldots, N.$$

Proof. Since $n_j^+ < n_j^-$, there are $N - j$ numbers of n_k^+ and $N - j + 1$ of n_k^-, which are larger than n_j^+, that is, we have

$$(N-j) + (N-j+1) \leqslant 2N - n_j^+.$$

This implies the lemma. □

In terms of the Young diagram Y^+, we have

Corollary 2.1. *The maximum diagram for* Y^+*, denoted as* $Y_{\max}$*, is the Young diagram associated with* $\mathbf{n}^+ = (1, 3, \ldots, 2N-1)$*, that is,* $Y_{\max}$ *is the upside down staircase with the size*

$$|Y_{\max}| = \frac{N(N-1)}{2}.$$

The maximum Young diagram for Y^- *is also given by* $Y_{\max}$*, which now corresponds to the case* $\mathbf{n}^- = (2N, 2N-1, \ldots, N+1)$*. Then the Young diagrams* $Y^\pm$ *associated with N-soliton solutions are given by a subdiagram of* $Y_{\max}$*.*

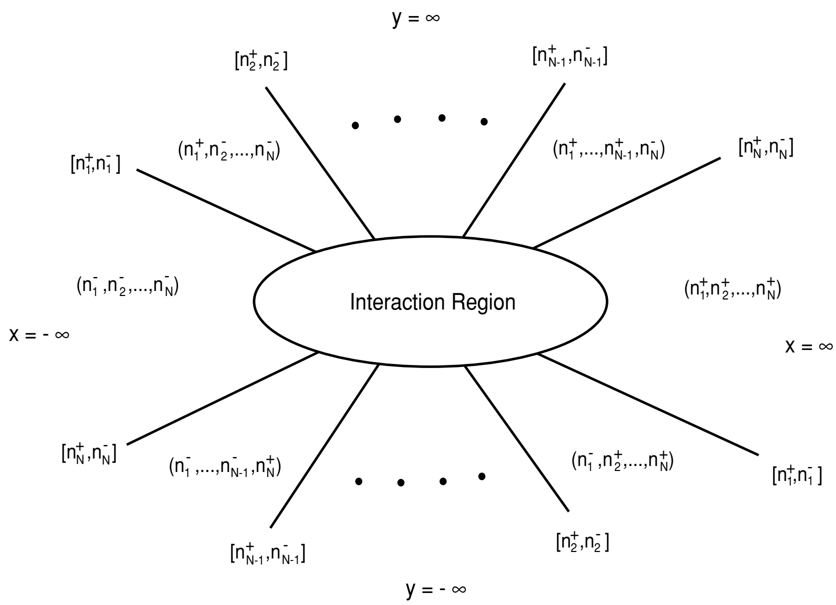

Figure 1. An *N*-soliton solution. The label $[n_j^+, n_j^-]$ indicates the $[n_j^+, n_j^-]$-soliton. Here the soliton velocities $c_j := k_{n_j^+} + k_{n_j^-}$ are assumed to be ordered as $c_{j+1} > c_j$. Each asymptotic region is marked by the value of the function w there, i.e. $(i_1, \ldots, i_N)$ implies $w = \sum_{j=1}^N k_{i_j}$.

Note that $|Y_{\max}|$ is also the number of intersection vertices of N lines in a plane in the general position. We will then determine whether each point is of resonant or non-resonant type.

In figure 1, we illustrate the asymptotic stage of an N-soliton solution with N solitons marked by $\left[n_j^+, n_j^-\right]$ for $j = 1, \ldots, N$. We note a duality of the values of w in the sense that if w takes $\sum_{j=1}^N k_{i_j}$, denoted as $(i_1, \ldots, i_N)$ in the figure, then w also takes $(j_1, \ldots, j_N)$ which is the complementary set of $(i_1, \ldots, i_N)$ in $\{1, \ldots, 2N\}$. In terms of the minors, this implies that if $\xi(i_1, \ldots, i_N) \neq 0$ then $\xi(j_1, \ldots, j_N) \neq 0$. The duality can be considered as a geometric symmetry of the N-soliton solution: in particular, if $\theta_j^0 = 0$ for all j, then N-soliton solution has the symmetry $(x, y, t) \leftrightarrow (-x, -y, -t)$. In this case, at $t = 0$ all the solitons intersect at the origin in the x–y plane (see figure 4). Then each soliton can be moved by shifting θ_j^0. A duality in the interaction region can be seen in figure 2 where two figures ((*A*) and (*B*)) of 3-soliton solution are illustrated. Note the duality in the middle region which implies $\xi\left(n_1^-, n_2^+, n_3^-\right) \neq 0$ and $\xi\left(n_1^+, n_2^-, n_3^+\right) \neq 0$. With this symmetry, we define the duality of the τ-function:

Definition 2.3. *We state that the τ-function in lemma 2.1 with $M = 2N$ satisfies the duality, if for any complementary sets $\{i_1, \ldots, i_N\}$ and $\{j_1, \ldots, j_N\}$, the minors satisfy*

$$\xi(i_1, \ldots, i_N) = 0 \quad \text{if and only if} \quad \xi(j_1, \ldots, j_N) = 0.$$

Before we discuss the general case with arbitrary N, we list up all possible 2-soliton solutions, and show that the τ-functions for those cases all satisfy the duality. As will be shown below, those solutions give the building blocks of N-soliton solutions. There are three cases of the solutions (see also [6]), which are labelled by the pair $(\mathbf{n}^+, \mathbf{n}^-)$ with $\mathbf{n}^\pm = \left(n_1^\pm, n_2^\pm\right)$. Two line solitons are labelled by $\left[n_1^+, n_1^-\right]$ and $\left[n_2^+, n_2^-\right]$:

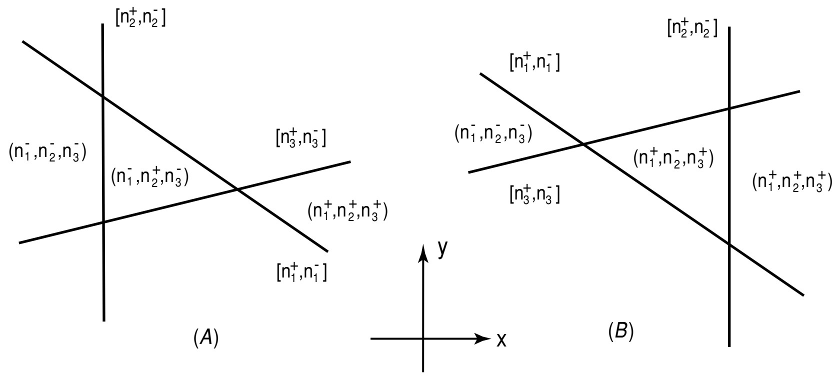

Figure 2. A duality in two equivalent 3-soliton solutions. The values of the function $w(x, y, t)$ in the middle region are expressed as $(n_1^{\pm}, n_2^{\mp}, n_3^{\pm})$ showing the duality of the τ-function.

(i) $\mathbf{n}^+ = (1, 3)$ and $\mathbf{n}^- = (2, 4)$. This corresponds to the ordinary 2-soliton solution, and the matrix A_2 which we denote as $A_2^{(O)}$, takes the form (after choosing θ_j^0 properly),

$$A_2^{(O)} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

which gives only four nonzero minors, $\xi(1, 3)$, $\xi(1, 4)$, $\xi(2, 3)$ and $\xi(2, 4)$. This shows the duality, i.e. $\xi(1, 2) = \xi(3, 4) = 0$. Those nonzero minors then label the four asymptotic regions separated by two line solitons: denoting $(i, j) := k_i + k_j$, w takes $(1, 3)$ as $x \to \infty$ which is dual to $(2, 4)$ as $x \to -\infty$. Also w takes $(1, 4)$ as $y \to \infty$, which is dual to $(2, 3)$ as $y \to -\infty$. This also implies that there is no other transition of the w value, hence the interaction is not of resonant type.

The Young diagram Y^+ in this case is the maximum one, i.e. $Y^+ = \square$. The diagram Y^- has no box, $Y^- = \emptyset$. The interaction of O-type is thus labelled by the Young diagram Y^+. It is also useful to note that the intervals defined by the labels $[1, 2]$ and $[3, 4]$ of solitons have no overlaps. Non-overlapping property can be stated as $\xi(1, 2) = 0$ and its dual one $\xi(3, 4) = 0$ (see also the examples in the end of section 4).

(ii) $\mathbf{n}^+ = (1, 2)$ and $\mathbf{n}^- = (3, 4)$. This case corresponds to the 2-soliton solution of the Toda lattice [1], and the matrix A_2, denoted as $A_2^{(T)}$, has the structure

$$A_2^{(T)} = \begin{pmatrix} 1 & 0 & - & - \\ 0 & 1 & + & + \end{pmatrix},$$

where '+, −' shows the signs of the entries (also nonzero). An explicit example of $A_2^{(T)}$ is

$$A_2^{(T)} = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

and the corresponding 2-soliton solution is illustrated as T-type in figure 3. Following the arguments in [1], one can easily show that two solitons are given by [1, 3]- and [2, 4]-solitons, and the interaction is in resonance. The main point in this example is that all the 2×2 minors of $A_2^{(T)}$ are nonzero, and this results in resonance interaction. The non-vanishing condition of all the minors is necessary to make the resonant interaction, and

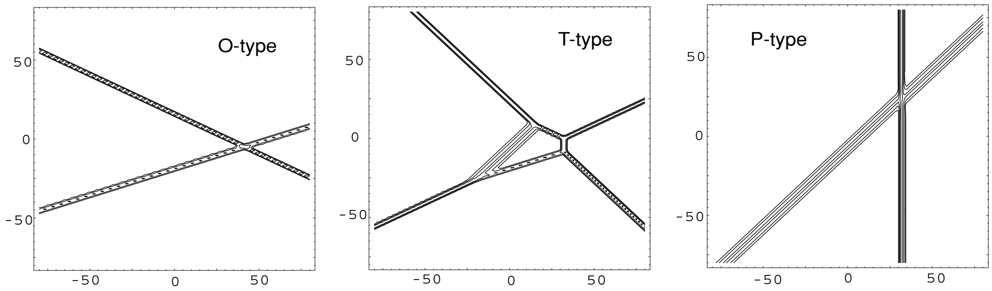

Figure 3. 2-soliton solutions. Two line solitons in those figures are [1, 2]- and [3, 4]-solitons for O-type, [1, 3]- and [2, 4]-solitons for T-type, and [1, 4]- and [2, 3]-solitons for P-type. The parameters are chosen as $(k_1, k_2, k_3, k_4) = (-2, 0, 1, 2)$ and $\theta_j^0 = 0, \forall j$ for all types.

those minors describe the intermediate solitons which form a Y-shape resonant interaction (see [1] for more detail).

Again we note the duality in the values of w in the asymptotics: w takes (3, 4) for $x \to -\infty$ which is dual to (1, 2) for $x \to \infty$. Also w takes (1, 4) for $y \to \infty$, which is dual to (2, 3) for $y \to -\infty$. One more duality appears in the inside of the resonant hole, that is, w takes (1, 3) for $t > 0$, which is dual to (2, 4) for $t < 0$ (or vice versa). Thus all six nonzero minors contribute to make the resonant interaction. The pair (Y^+, Y^-) of the Young diagrams is then given by (∅, ∅). Also note that the intervals defined by the labels of solitons, [1, 3] and [2, 4], have a partial overlap.

(iii) $\mathbf{n}^+ = (1, 2)$ and $\mathbf{n}^- = (4, 3)$. This case has been noted in [6], and the corresponding matrix A_2, denoted as $A_2^{(P)}$ (P stands for *physical*, see remark 2.4), is given by (after choosing θ_j^0 properly)

$$A_2^{(P)} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

which gives again only four nonzero minors with $\xi(1, 2), \xi(1, 3), \xi(2, 4)$ and $\xi(3, 4)$. This also implies no resonance, and the solitons are given by [1, 4]- and [2, 3]-solitons. This solution is similar to the O-type, but the pair of the Young diagrams is different, and given by (∅, □). The interaction of this type is labelled by the Young diagram Y^-. Also the intervals defined by the labels of solitons, [1, 4] and [2, 3], have complete overlap, i.e. the interval (2, 3) is in (1, 4). This implies $\xi(2, 3) = 0$ and its dual $\xi(1, 4) = 0$.

One should note that we have the following order in the soliton velocities $c_{ij} := k_i + k_j$ among those solitons for given constants k_j with the order $k_1 < k_2 < k_3 < k_4$,

$$c_{12} < c_{13} < c_{14}, c_{23} < c_{24} < c_{34}.$$

Here note that c_{14} and c_{23} cannot have a definite order (even $c_{14} = c_{23}$ is possible). This ordering indicates that any 2-soliton solution can be classified as one of those three solitons. One should note here that the type for a given pair of solitons is completely determined by their labels $\left[n_j^+, n_j^-\right]$ for $j = 1, 2$. Namely, the interaction pattern is completely determined by their asymptotics in $y \to \infty$, where two solitons are labelled.

Although the interaction patterns of the cases of O-type in (ii) and P-type in (iii) are the same, i.e. non-resonant, they describe orbits on different cells in terms of the Schubert decomposition of the Grassmannian, in which each cell is parametrized by a Young diagram. We will give more details of this description in section 3.

Remark 2.4. In [7], the KP equation was introduced to describe a transversal stability of the KdV soliton propagating along the x-axis. Then the order of the wavenumber in the y-direction is assumed to be much smaller than that in the x-direction. This implies that the wave described by the KP equation should be almost parallel to the y-axis for a better approximation within the physical setting. In this sense, the P-type solitons are more physical than other two types. The KdV two solitons are also obtained from the P-type solitons with $k_1 < k_2 < 0$ and $k_3 = -k_2, k_4 = -k_1$ (see section 5).

3. Grassmannian Gr(*N*, 2*N*) and *N*-soliton solutions

Here we briefly summarize the basics of the Grassmann manifold $\mathrm{Gr}(N, M)$ in order to explain that the minors $\xi(i_1, \ldots, i_N)$ in lemma 2.1 provide a coordinate system for $\mathrm{Gr}(N, M)$, the Plücker coordinates. Namely a solution given by the τ-function (1.1) can be marked by a point on $\mathrm{Gr}(N, M)$. The purpose of this section is to identify an N-soliton solution as a $2N$-dimensional torus orbit of a point on $\mathrm{Gr}(N, 2N)$ marked by the minors $\xi(i_1, \ldots, i_N)$. Then using the Schubert decomposition of the Grassmannian $\mathrm{Gr}(N, 2N)$, we classify the orbits which represent the N-soliton solutions.

3.1. Grassmannian Gr(N, M)

A real Grassmannian $\mathrm{Gr}(N, M)$ is the set of N-dimensional subspaces of $\mathbb{R}^M$. A point ξ of the Grassmannian is expressed by the N-frame of vectors,

$$\xi = [\xi_1, \xi_2, \ldots, \xi_N], \qquad \text{with} \quad \xi_i = \sum_{j=1}^{M} a_{ij} e_j \in \mathbb{R}^M,$$

where $\{e_i \,|\, i = 1, 2, \ldots, M\}$ is the standard basis of $\mathbb{R}^M$, and $(a_{ij}) = A_{(N,M)}$ is the $N \times M$ matrix given in (1.2). Then the Grassmannian $\mathrm{Gr}(N, M)$ can be embedded into the projectivization of the exterior space $\bigwedge^N \mathbb{R}^M$, which is called the Plücker embedding,

$$\begin{aligned} \mathrm{Gr}(N, M) &\hookrightarrow \mathbb{P}\left(\textstyle\bigwedge^N \mathbb{R}^M\right) \\ \xi = [\xi_1, \ldots, \xi_N] &\mapsto \xi_1 \wedge \cdots \wedge \xi_N \end{aligned}$$

Here the element on $\mathbb{P}\left(\bigwedge^N \mathbb{R}^M\right)$ is expressed as

$$\xi_1 \wedge \cdots \wedge \xi_N = \sum_{1 \leqslant i_1 < \cdots < i_N \leqslant M} \xi(i_1, \ldots, i_N) e_{i_1} \wedge \cdots \wedge e_{i_N},$$

where the coefficients $\xi(i_1, \ldots, i_N)$ are $N \times N$ minors given in lemma 2.1, which are called the Plücker coordinates.

It is also well known that the Grassmannian can have the cellular decomposition, called the Schubert decomposition [4],

$$\mathrm{Gr}(N, M) = \bigsqcup_{1 \leqslant i_1 < \cdots < i_N \leqslant M} W(i_1, \ldots, i_N) \tag{3.1}$$

where the cells are defined by

$$W(i_1,\ldots,i_N) = \left\{ \xi = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ * & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ * & * & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{pmatrix} \in \overbrace{\mathbb{R}^M \times \cdots \times \mathbb{R}^M}^{N} \right\}.$$

$$= \{\text{the matrix } \xi = A^T_{(N,M)} \text{ in the echelon form whose pivot ones are at } (i_1,\ldots,i_N) \text{ positions}\}$$

Namely, an element $\xi = [\xi_1,\ldots,\xi_N] \in W(i_1,\ldots,i_N)$ is described by

$$\xi \in W(i_1,\ldots,i_N) \quad \Leftrightarrow \quad \begin{cases} \text{(i)} \quad \xi(i_1,\ldots,i_N) \neq 0. \\ \text{(ii)} \quad \xi(j_1,\ldots,j_N) = 0 \quad \text{if } j_n < i_n \text{ for some } n. \end{cases}$$

Each cell $W(i_1,\ldots,i_N)$ is called the Schubert cell, and it is convenient to label the cell by the Young diagram $Y = (i_1,\ldots,i_k)$ where the number of boxes is given by $\ell_j = i_j - j$ for $j = 1,\ldots,N$ (counted from the bottom), which also expresses a partition $(\ell_N, \ell_{N-1},\ldots,\ell_1)$ of the number $|Y| := \sum_{i=1}^N \ell_i$, the *size* of Y. The codimension of the cell $W(i_1,\ldots,i_N)$ is then given by the size of Y, i.e.,

$$\operatorname{codim} W(i_1,\ldots,i_N) = |Y| = (i_1 + \cdots + i_N) - \tfrac{1}{2}N(N+1),$$

and the dimension is given by the number of free variables in the echelon form. Note that the top cell of $\mathrm{Gr}(N,M)$ is labelled by $Y = (1,\ldots,N)$, i.e. $|Y| = 0$, and

$$\dim W(1,\ldots,N) = \dim \mathrm{Gr}(N,M) = N \times (M-N).$$

The main point here is that each matrix $A_{(N,M)}$ in (1.2) can be identified as a point on a Schubert cell of the Grassmannian. Now we identify an N-soliton solution as a torus orbit of a point on the Grassmannian $\mathrm{Gr}(N,2N)$, and show that orbits in the different cells give different types of N-soliton solutions.

3.2. N-soliton solution as a 2N-dimensional torus orbit

First we note that the τ-function (1.1) in the Wronskian form can be expressed as

$$\tau(x,y,t) = \langle v^N, KD(x,y,t)A^t \cdot v^N\rangle, \tag{3.2}$$

where v^N is the highest weight vector in a fundamental representation of $\mathrm{GL}(2N)$ which is given by the element of $\bigwedge^N \mathbb{R}^{2N}$,

$$v^N = e_1 \wedge e_2 \wedge \cdots \wedge e_N,$$

and K, A (A^t represents the transpose of A) are $2N \times 2N$ constant matrices defined by

$$\begin{cases} K = (k_{ij})_{1\leqslant i,j\leqslant 2N}, & \text{with} \quad k_{ij} := (-k_j)^{i-1} \\ A = (a_{ij})_{1\leqslant i,j\leqslant 2N}. \end{cases}$$

The $D(x, y, t)$ is $2N \times 2N$ diagonal matrix associated with an $(\mathbb{R}^*)^{2N}$-torus action,

$$D(x, y, t) = \mathrm{diag}(\mathrm{e}^{\theta_1}, \ldots, \mathrm{e}^{\theta_{2N}}) \qquad \text{with} \quad \theta_j = -k_j x + k_j^2 y - k_j^3 t + \theta_j^0.$$

Note here that $A^t \cdot v^N$ defines the matrix $A_N = A_{(N,2N)}$ in (1.2), that is,

$$A^t \cdot e_1 \wedge \cdots \wedge e_N = A^t e_1 \wedge A^t e_2 \wedge \cdots \wedge A^t e_N,$$

which can be identified as A_N^t. The inner product $\langle \cdot, \cdot \rangle$ on $\bigwedge^N \mathbb{R}^{2N}$ is defined by

$$\langle v_1 \wedge \cdots \wedge v_j, w_1 \wedge \cdots \wedge w_j \rangle = \det[(\langle v_m, w_n \rangle)_{1 \leqslant m,n \leqslant j}],$$

where $\langle v_m, w_n \rangle$ is the standard inner product of $v_m, w_n \in \mathbb{R}^{2N}$.

The expression (3.2) implies that the solution $u(x, y, t)$ given by the τ-function (1.1) is a $2N$-dimensional torus orbit, $(\mathbb{R}^*)^{2N}$-orbit, of a point on $\mathrm{Gr}(N, 2N)$ marked by A_N. Note also that the Schubert cell $W(i_1, \ldots, i_N)$ is invariant under this torus action. The orbits, N-soliton solutions, can be first classified in terms of the Schubert decomposition with the index sets $(i_1, \ldots, i_N)$.

Example 3.1 Gr(2, 4)**.** The Schubert decomposition of Gr(2, 4) is given by

$$\mathrm{Gr}(2, 4) = \bigsqcup_{1 \leqslant i, j \leqslant 4} W(i, j).$$

There are six cells $W(i, j)$ with $\dim W(i, j) = 7 - (i + j)$. (Note $7 = N(M - N) + N(N + 1)/2$ with $N = 2$, $M = 4$.)

(i) $W(1, 2)$. This is a top cell of maximum dimension 4, and a point on the cell is given by

$$A_2 = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix},$$

which includes 2-soliton solutions of T-type, $A_2^{(T)}$, and P-type, $A_2^{(P)}$.

(ii) $W(1, 3)$. A point on this cell is marked by

$$A_2 = \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}.$$

The 2-soliton solution of O-type with $A_2^{(O)}$ belongs to this cell. Thus O-type and P-type belong to different cells, even though they show non-resonant interactions.

(iii) $W(1, 4)$. The A_2 matrix has the form

$$A_2 = \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case, nonzero minors are given by $\xi(i, 4)$ for $i = 1, 2, 3$. This implies that the exponential term $E_4 = \mathrm{e}^{\theta_4}$ becomes a common factor in the τ-function. Then the solution gives (2, 1)-soliton solution, that is, two incoming solitons and one outgoing soliton in the asymptotics $y \to \pm\infty$ (see [1]).

(iv) $W(2, 3)$. The A_2 matrix is given by

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}.$$

Then the exponential term $E_1 = \mathrm{e}^{\theta_1}$ is missing in the τ-function, and the solution describes a (1, 2)-soliton solution.

(v) $W(2,4)$. The A_2 matrix is

$$A_2 = \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The solution is a 1-soliton solution of [2, 3]-soliton.

(vi) $W(3,4)$. The A_2 matrix is

$$A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The τ-function is just a product of $E_3 = \mathrm{e}^{\theta_3}$ and $E_4 = \mathrm{e}^{\theta_4}$, and the solution is trivial, $u = 0$.

Remark 3.2. Since $W(1,3)$ is a boundary of the top cell $W(1,2)$, a point on $W(1,3)$ can be obtained by a limit of a sequence of points in $W(1,2)$. For example, consider the following matrix (representing one-parameter family of points in $W(1,2)$),

$$A_2(\epsilon) = \begin{pmatrix} 1 & 0 & -\epsilon^{-1} & -\epsilon^{-1} \\ 0 & 1 & \epsilon^{-1}+\epsilon & \epsilon^{-1}-\epsilon \end{pmatrix},$$

where the parameter ϵ is a positive number. If $\epsilon < 1$, then the matrix gives a 2-soliton solution of T-type (note that all the minors are positive). Then in the limit $\epsilon \to 0$, the matrix can be interpreted as the O-type matrix $A_2^{(O)}$ in $W(1,3)$ (recall here that $\epsilon^{-1}\tau \equiv \tau$ is the Plücker coordinate, i.e. they both give the same solution). It is also obvious that $A_2^{(P)}$ is obtained by a limit of the T-type matrix. Thus T-type is generic, while the O-type and P-type are non-generic. Also in the Gallipoli conference in June–July 2004, Pashaev showed that a 2-soliton solution of T-type is a degenerate case of 4-soliton solution of non-resonant type (obtained by the Hirota method) [8].

Since *N*-soliton solution is an orbit on a particular Schubert cell $W(i_1, \ldots, i_N)$ which is a boundary of the top cell $W(1, \ldots, N)$, it seems to be true that all the *N*-soliton solutions are obtained by the limits of the τ-function of T-type which lives on the top cell. We will discuss more details in a future communication.

4. Construction of the A_N matrix

In this section, we give an explicit construction of the matrix A_N for *N*-soliton solution parametrized by the pair $(\mathbf{n}^+, \mathbf{n}^-)$: the key idea of the construction is based on the classification of 2-soliton solutions, that is, a local structure of the matrix contains one of those types, O-, T- or P-type. With a given pair $(\mathbf{n}^+, \mathbf{n}^-)$, we construct an *N*-soliton solution consisting of *N* line solitons with the labels $[n_j^+, n_j^-]$ for $j = 1, \ldots, N$. In order to do this, let us first define

Definition 4.1. *The length of* $[n_j^+, n_j^-]$*-soliton, denoted by* L_j*, is defined by*

$$L_j := \min\{n_j^- - n_j^+, 2N - n_j^- + n_j^+\}.$$

Note that $1 \leqslant L_j \leqslant N$ *for all* $j = 1, \ldots, N$.

In the cases of 2-soliton solutions, we have

(i) for O-type, $L_1 = L_2 = 1$;
(ii) for T-type, $L_1 = L_2 = 2$;
(iii) for P-type, $L_1 = L_2 = 1$.

Then we note that the length of the line soliton gives the condition on the minors. In those cases, $\xi\left(n_j^+, n_j^-\right) = 0$ if and only if $L_j < 2$.

For N-soliton solutions with $\left[n_j^+, n_j^-\right]$-solitons, we impose that the τ-functions satisfy the following condition which plays a crucial role for the construction of A_N:

Definition 4.2. *We say that a τ-function satisfies the N-soliton condition, if it satisfies the duality (definition 2.3) and has missing exponential terms (i.e. vanishing minors) if and only if there exists a $\left[n_j^+ n_j^-\right]$-soliton whose length is less than N. If $L_j < N$ for some j, we require that*

- *for $L_j = n_j^- - n_j^+$, then the $N \times N$ minors satisfy*

$$\xi\left(\ldots, \left[n_j^+, n_j^+ + 1, \ldots, n_j^-\right], \ldots\right) = 0,$$

- *for $L_j = 2N - n_j^- + n_j^+$, then*

$$\xi\left(\left[1, \ldots, n_j^+\right], \ldots, \left[n_j^-, \ldots, 2N\right]\right) = 0.$$

Here the $L_j + 1$ entries inside the brackets $[\cdots]$ are consecutive numbers, and the other entries in the '...' outside the $[\cdots]$ can take any numbers.

Recall that in the case of N-soliton solutions for the Toda lattice, every soliton has the length N, and there is no vanishing minor [1]. The N-soliton condition is a natural extension of that in the case of 2-soliton solutions, and it provides local information of 2-soliton interactions in an N-soliton solution which are classified into three types, O, T and P. Note in particular, with this condition, that one identifies all the vanishing minors in the τ-function.

Let us now construct the matrix A_N: the construction has several steps, and it might be better to explain the steps using an explicit example.

Let us consider the case $N = 4$ with $\mathbf{n}^+ = (1, 2, 4, 5)$ and $\mathbf{n}^- = (3, 7, 6, 8)$:

Step I. Since the minors $\xi(1, 2, 4, 5) \neq 0$ and $\xi(3, 6, 7, 8) \neq 0$, one can easily see that A_N matrix has the structure

$$\begin{pmatrix} * & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \end{pmatrix}, \tag{4.1}$$

where '$*$' marks a possible nonzero entry, and in particular, the first element in each row should be nonzero (this is the pivot place). Note here that $\xi(1, 2, 3, j) = 0$ for any j, which is also considered as the 4-soliton condition with the length $L_1 = 2$ of the [1, 3]-soliton. Note that $\xi(i_1, i_2, i_3, i_4) = 0$ for any $4 \leqslant i_1, \ldots, i_4 \leqslant 8$ are dual to $\xi(1, 2, 3, j) = 0$ for some j.

Step II. In this step, we put the form (4.1) into the RREF. First we normalize the first nonzero entries to be 1 in every row, and eliminate the (2, 4)-entry from the third row and the (3, 5)-entry from the fourth row. Note that this process does not affect the 0s of the matrix in (4.1). Then we have

$$\begin{pmatrix} 1 & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & * & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * \end{pmatrix}. \tag{4.2}$$

Further eliminating '$*$' in the second column by row reduction using the 2nd row, we have

$$\begin{pmatrix} 1 & 0 & * & 0 & 0 & \# & \# & \# \\ 0 & 1 & * & 0 & 0 & \# & \# & \# \\ 0 & 0 & 0 & 1 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * \end{pmatrix}, \tag{4.3}$$

where the two rows in the 2×3 submatrix with # entries are parallel (recall $\xi(5,6,7,8) = \xi(4,6,7,8) = 0$). Also note that the signs of nonzero entries are uniquely determined by the requirement of all the minors to be non-negative; for example, the last three entries in the first row are non-positive, and the signs of the last three entries of the following rows change alternatively. Up to this point, we have not used the order in $\mathbf{n}^-$. In another words, the form of the matrix in (4.3) is common for any $\mathbf{n}^-$ with a given $\mathbf{n}^+$.

Step III. Now we use the orders in $\mathbf{n}^-$ to determine explicit entries in the matrix for a 4-soliton solution consisting of four line solitons labelled by $[n_j^+, n_j^-]$, i.e. [1, 3], [2, 7], [4, 6], [5, 8]. In this step, we identify all other zero elements (i.e. zero minors) of the matrix A_N using the *N*-soliton condition (definition 4.2). The following graph shows the overlapping relations among those solitons and the length of each soliton:

$$\begin{array}{llllllllll} \circ & - & \bullet & & & & & & & : L_1 = 2 \\ & \circ & - & - & - & - & \bullet & & & : L_2 = 3 \\ & & & \circ & - & \bullet & & & & : L_3 = 2 \\ & & & & \circ & - & - & \bullet & & : L_4 = 3. \end{array} \tag{4.4}$$

Now we use the 4-soliton condition. First we note that from $L_1 = 2 < 4$ we have $\xi(1,2,3,j) = 0$ for any j (see step I). Also from $L_3 = 2$, we have $\xi(4,5,6,j) = 0$ for any j. In particular, $\xi(1,4,5,6) = 0$ implies that $(2,6)$-entry must be zero, so that $(1,6)$-entry is also zero. Also $\xi(3,4,5,6) = 0$ gives $\xi(1,2,7,8) = 0$ which is also the 4-soliton condition with $L_2 = 3$. The $\xi(1,2,7,8) = 0$ implies that the 2×2 determinant of the right bottom corner must be zero. We then get the following structure,

$$\begin{pmatrix} 1 & 0 & * & 0 & 0 & 0 & \# & \# \\ 0 & 1 & * & 0 & 0 & 0 & \# & \# \\ 0 & 0 & 0 & 1 & 0 & * & \times & \times \\ 0 & 0 & 0 & 0 & 1 & * & \times & \times \end{pmatrix},$$

where the 2×2 minor with the $\times$ entries is zero. Also from the partial overlaps (i.e. T-type interaction) in [1, 3] with [2, 7], in [2, 7] with [5, 8] and in [4, 6] with [5, 8], we need nonzero entries for those marked by $*$, # and $\times$. Then one can easily find an explicit example of the matrix A_4 whose 4×4 minors are all non-negative,

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & -3 & -9 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & -1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

In figure 4, we illustrate a 4-soliton solution generated with this matrix. Note that there are three resonant vertices corresponding to the interactions between [1, 3] and [2, 7], [2, 7] and [5, 8], and [4, 6] and [5, 8] (see the overlapping graph in (4.4)). Other three vertices are two of O-type and one P-type.

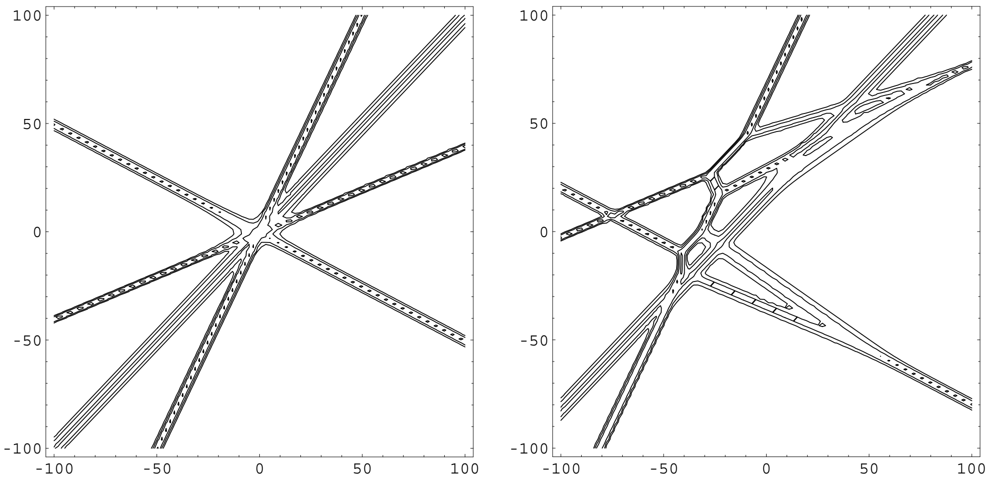

Figure 4. 4-soliton solutions labelled by $\mathbf{n}^+ = (1, 2, 4, 6)$ and $\mathbf{n}^- = (3, 7, 6, 8)$. The left figure shows the 4-soliton solution at $t = 0$ and the right one at $t = 10$. The parameters are given by $\theta_j^0 = 0$ for all $j = 1, \ldots, 8$, and $(k_1, \ldots, k_8) = (-3/2, -1, -1/2, 0, 1/2, 1, 3/2, 2)$. The solution at $t = -10$ can be obtained by the reflection $(-x, -y)$ of the right graph.

One can show the following proposition for the general case:

Proposition 4.1. *If there is a pair* $[n_j^+, n_j^-]$ *with* $L_j = n_j^- - n_j^+ < N$*, then the* $N \times (L_j + 1)$ *submatrix* (a_{mn}) *with* $1 \leqslant m \leqslant N$ *and* $n_j^+ \leqslant n \leqslant n_j^-$ *of the matrix* A_N *has the maximum rank* L_j*. (The same is also true for the case* $L_j = 2N - n_j^- + n_j^+$*.)*

Proof. Since $L_j < N$, we have $\xi\left([n_j^+, \ldots, n_j^-], *, \ldots, *\right) = 0$ (see the *N*-soliton condition). Here the entries marked with $*$ are arbitrary columns of A_N whose maximum rank is *N*. If we choose $N - L_j - 1$ independent columns for those entries, then the $\xi = 0$ implies that the submatrix can have at most the rank L_j. □

Then the following is immediate as a corollary of this proposition:

Corollary 4.1. *If a pair* $[n_j^+, n_j^-]$ *has the minimum length, i.e.* $n_j^- = n_j^+ + 1$*, then the* $N \times 2$ *submatrix with the columns* (a_{mn}) *with* $1 \leqslant m \leqslant N$ *and* $n = n_j^+, n_j^-$ *of the matrix should be in the form,*

$$\begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \quad \text{where 1s are in the } j\text{th row.}$$

Proof. From proposition 4.1, the rank of the submatrix is 1. This implies that the rows of the submatrix are all proportional. Since the entry (j, n_j^+) is the pivot 1, all the other entries of this column should be zero in the RREF. This gives the result. □

This corollary shows that the line soliton marked by $\left[n_j^+, n_j^-\right]$ of length 1 intersects with all other solitons without resonance, that is, the interactions are either O-type or P-type. Note that in this case $\left[n_j^+, n_j^-\right]$ cannot have partial overlap with other pairs, and this can be seen in the matrix A_N. Proposition 4.1 provides the interaction structure of *N*-soliton solution for O-type and P-type.

We now show that the Young diagrams Y^+ and Y^- provide some information on the interaction pattern of the *N*-soliton solution. If we ignore the phase shifts and resonances among the solitons, the *N*-soliton solution has $\frac{N(N-1)}{2}$ vertices in a generic situation. Then one can find the number of vertices having a particular type of interactions.

Theorem 4.3. *The N-soliton solution labelled by the pair of Young diagrams* (Y^+, Y^-) *has*

- $|Y^+|$ *number of vertices of O-type, and*
- $|Y^-|$ *number of vertices of P-type.*

The number of resonant vertices, T-type, is then given by

$$\frac{N(N-1)}{2} - |Y^+| - |Y^-|.$$

Proof. The Young diagram Y^+ can be obtained directly from the matrix A_N: for example, consider $\mathbf{n}^+ = (1, 2, 4, 5, 7)$ and $\mathbf{n}^- = (3, 9, 6, 10, 8)$. Then the graph showing the overlaps among $\left[n_j^+, n_j^-\right]$ is given by

$$\begin{array}{cccccccccc}
\circ & - & \bullet & & & & & & & \\
 & \circ & - & - & - & - & - & - & \bullet & \\
 & & \square & \circ & - & \bullet & & & & \\
 & & \square & & \circ & - & - & - & - & \bullet \\
 & & \square & & & \square & \circ & \bullet & &
\end{array}$$

where the boxes under the non-pivots ($\bullet$) give Y^+, which is given by $(1, 1, 2)$ representing the number of boxes in the row from the bottom (Y^+-shape is just upside down picture of one given in the graph). It is then obvious that each $\square$ provides O-type interaction, i.e. two solitons having no overlap. For example, $[1, 3]$ has no overlap with $[4, 6]$, $[5, 10]$ and $[7, 8]$. This proves that $|Y^+|$ gives the number of vertices of O-type.

The diagram Y^- gives the number of *reverses* in the sequence $(n_1^-, \ldots, n_N^-)$. Again consider the example above. The number $n_2^- = 9$ has two reverses, 6 and 8. Then from the graph above, this implies that $[2, 9]$ has complete overlap with $[4, 6]$ and $[7, 8]$, i.e. $[2, 9]$-soliton has P-type interaction with those solitons. Now it is obvious that $|Y^-|$ gives the number of vertices of P-type. $\square$

Before ending this section, we give a complete list of the 3-soliton solutions with explicit examples of A_3 matrices:

Case a: $\mathbf{n}^+ = (1, 2, 3)$. In this case we have $m_{\mathbf{n}^+} = 1 \cdot 2 \cdot 3 = 6$ different choices of $\mathbf{n}^-$; that is, six different 3-soliton solutions. Since the Young diagram Y^+ has no box, $Y^+ = \emptyset$, there is no O-type vertex in those solutions.

a1: $\mathbf{n}^- = (4, 5, 6)$. This is the case of the Toda lattice [1]. The pair of Young diagrams is $(\emptyset, \emptyset)$, that is, all the vertices are T-type. Example of the matrix A_3 is given by the following matrix. We also show the overlapping graph:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 3 & 6 \\ 0 & 1 & 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \qquad \begin{array}{cccccc}
\circ & - & - & \bullet & & \\
 & \circ & - & - & \bullet & \\
 & & \circ & - & - & \bullet
\end{array}$$

a2: $\mathbf{n}^- = (4, 6, 5)$. This has two T-type vertices and one P-type vertex, i.e. $Y^- = \square$. In particular, the P-type interaction between [2, 6]- and [3, 5]-solitons implies that the (3, 6)-entry should be zero (because of $\xi(1, 2, 6) = 0$). Then we have an example

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \qquad \begin{array}{cccccc} \circ & - & - & \bullet & & \\ & \circ & - & - & - & \bullet \\ & & \circ & - & \bullet & \end{array}$$

Note here that the duality of $\xi(1, 2, 6) = 0$ and $\xi(3, 4, 5) = 0$. Both vanishing minors are due to the 3-soliton condition for $L_2 = L_3 = 2$.

a3: $\mathbf{n}^- = (5, 4, 6)$. This has again two T-type vertices and one P-type vertex (note that the Young diagram Y^- is the same as case 2a). An example of A_3 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & -2 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \qquad \begin{array}{cccccc} \circ & - & - & - & \bullet & \\ & \circ & - & \bullet & & \\ & & \circ & - & - & \bullet \end{array}$$

Again note the 3-soliton conditions for $L_1 = L_2 = 2$ which give $\xi(1, 5, 6) = 0$ and its dual $\xi(2, 3, 4) = 0$. The $\xi(2, 3, 4)$ then gives (1, 4)-entry to be zero.

a4: $\mathbf{n}^- = (5, 6, 4)$. There are two P-type and one T-type vertices. We have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \qquad \begin{array}{cccccc} \circ & - & - & - & \bullet & \\ & \circ & - & - & - & \bullet \\ & & \circ & \bullet & & \end{array}$$

Note here that the [3, 4]-soliton has a P-type interaction with other two solitons, which gives the last row to be (0, 0, 1, 1, 0, 0) and third and fourth columns to be $e_3 \in \mathbb{R}^3$ (see corollary 4.1).

a5: $\mathbf{n}^- = (6, 4, 5)$. This case is similar to the previous one, i.e. two P-types and one T-type. A_3 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \qquad \begin{array}{cccccc} \circ & - & - & - & - & \bullet \\ & \circ & - & \bullet & & \\ & & \circ & - & \bullet & \end{array}$$

a6: $\mathbf{n}^- = (6, 5, 4)$. All the vertices are of P-type, and the matrix A_3 can be written as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \qquad \begin{array}{cccccc} \circ & - & - & - & - & \bullet \\ & \circ & - & - & \bullet & \\ & & \circ & \bullet & & \end{array}$$

The Young diagram Y^- is maximum with three boxes.

In these examples, one should note that the pattern of the zeros in the last three columns (non-pivot columns) presents the shape of the corresponding Young diagram Y^-.

Case b: $\mathbf{n}^+ = (1, 2, 4)$. There are $m_{\mathbf{n}^+} = 1 \cdot 2 \cdot 2 = 4$ different types of 3-soliton solutions. Now the Y^+ has one box, there is one O-type interaction in these solutions.

b1: $\mathbf{n}^- = (3, 5, 6)$. There are two T-type vertices and one O-type. An example of A_3 matrix is

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \qquad \begin{array}{cccccc} \circ & - & \bullet & & & \\ & \circ & - & - & \bullet & \\ & & & \circ & - & \bullet \end{array}$$

Note here the duality in $\xi(1, 2, 3) = 0$ and $\xi(4, 5, 6) = 0$.

b2: $\mathbf{n}^- = (3, 6, 5)$. The Y^- has one box, and the vertices are all different types; O, N and T. An example of A_3 can be

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \qquad \begin{array}{cccccc} \circ & - & \bullet & & & \\ & \circ & - & - & - & \bullet \\ & & & \circ & \bullet & \end{array}$$

b3: $\mathbf{n}^- = (5, 3, 6)$. This is similar to the previous case of b2. A_3 can be

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \qquad \begin{array}{cccccc} \circ & - & - & - & \bullet & \\ & \circ & \bullet & & & \\ & & & \circ & - & \bullet \end{array}$$

b4: $\mathbf{n}^- = (6, 3, 5)$. The Y^- has two boxes, and there are two P-type vertices and one O-type. The matrix A_3 is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \qquad \begin{array}{cccccc} \circ & - & - & - & - & \bullet \\ & \circ & \bullet & & & \\ & & & \circ & \bullet & \end{array}$$

Case c: $\mathbf{n}^+ = (1, 2, 5)$. There are $m_{\mathbf{n}^+} = 1 \cdot 2 \cdot 1 = 2$ different 3-soliton solutions. Since the Y^+ has two horizontal boxes, there are two O-type vertices in these solutions.

c1: $\mathbf{n}^- = (3, 4, 6)$. The Y^- has no box, and the 3-soliton solution has one T-type interaction with two O-types. A matrix A_3 can be

$$\begin{pmatrix} 1 & 0 & -1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \qquad \begin{array}{cccccc} \circ & - & \bullet & & & \\ & \circ & - & \bullet & & \\ & & & & \circ & \bullet \end{array}$$

c2: $\mathbf{n}^- = (4, 3, 6)$. The Y^- has one box, and the solution has one P-type interaction with two O-types. The matrix A_3 is given by

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \qquad \begin{array}{cccccc} \circ & - & - & \bullet & & \\ & \circ & \bullet & & & \\ & & & & \circ & \bullet \end{array}$$

Case d: $\mathbf{n}^+ = (1, 3, 4)$. The Y^+ has two boxes, and there are $m_{\mathbf{n}^+} = 1 \cdot 1 \cdot 2 = 2$ different 3-soliton solutions which have two O-type interactions:

d1: $\mathbf{n}^- = (2, 5, 6)$. Since $Y^- = \emptyset$, there is one T-type intersection with two O-types. An example of the matrix A_3 is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \qquad \begin{array}{cccccc} \circ & \bullet & & & & \\ & & \circ & - & \bullet & \\ & & & \circ & - & \bullet \end{array}$$

d2: $\mathbf{n}^- = (2, 6, 5)$. Now Y^- has one box, and the solution has one P-type interaction with two O-types. The matrix A_3 is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \qquad \begin{array}{cccccc} \circ & \bullet & & & & \\ & & \circ & - & - & \bullet \\ & & & \circ & \bullet & \end{array}$$

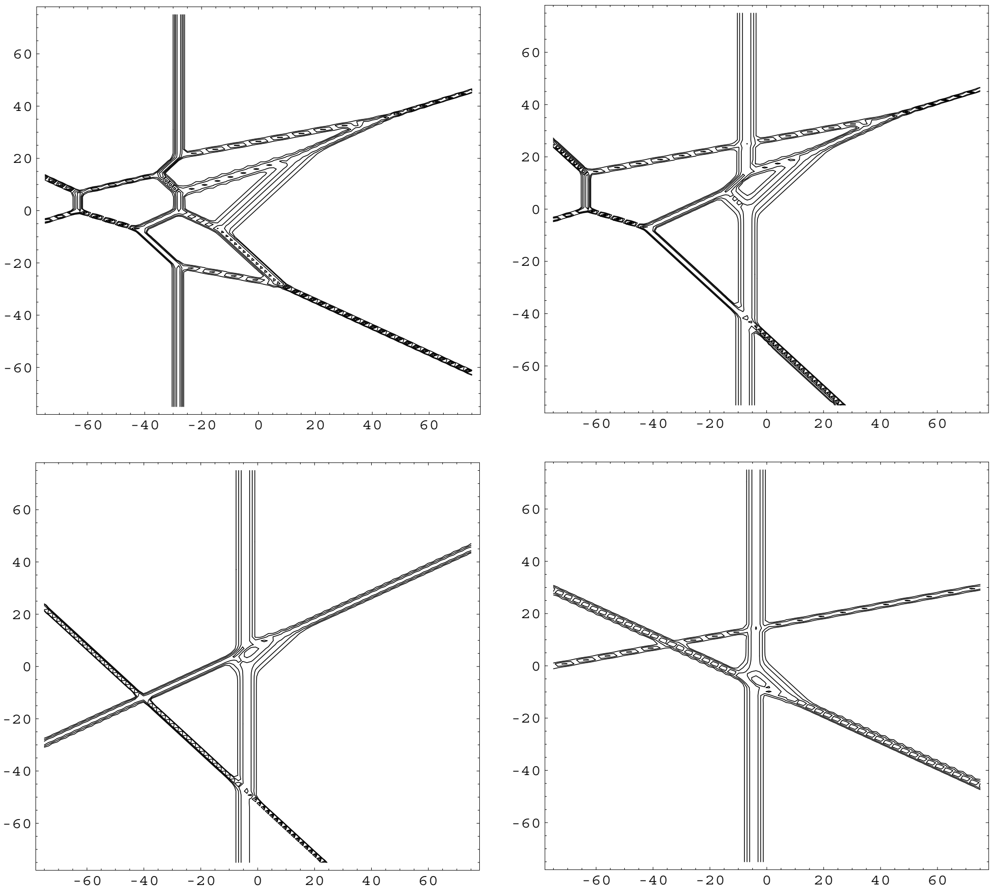

Figure 5. 3-soliton solutions. All the cases assume $\theta_j^0 = 0$ for all $j = 1, \ldots, 6$. The top left figure shows the solution in case a1 with the parameters $(k_1, \ldots, k_6) = (-3, -2, 0, 1, 2, 3)$ and $t = 7$. The top right figure shows case a3 with $(-3, -1, 0, 1, 2, 3)$ and $t = 7$. The one in the bottom left shows case a5 with $(-4, -1, 0, 1, 2, 3)$ and $t = 4$, and that in the bottom right shows case c1 with $(-2, -1, 0, 1, 2, 3)$ and $t = 4$.

Case e: $\mathbf{n}^+ = (1, 3, 5)$. This is the ordinary 3-soliton solution with the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \qquad \begin{matrix} \circ & \bullet & & & & \\ & & \circ & \bullet & & \\ & & & & \circ & \bullet \end{matrix}$$

Some of the 3-soliton solutions are illustrated in figure 5. Note that in the matrix A_3, if there is only one nonzero entry in a column (like a pivot), then one can set the entry to be +1 or -1 by choosing some phase constant θ_j^0. Those A_3 in cases a6, b4, c2, d2 and e are unique in this sense.

5. *N*-soliton solutions of the KdV equation

Here we consider *N*-soliton solutions of the KdV equation, and show that they cannot have a resonant interaction. *N*-soliton solutions of the KdV equation can be obtained by the

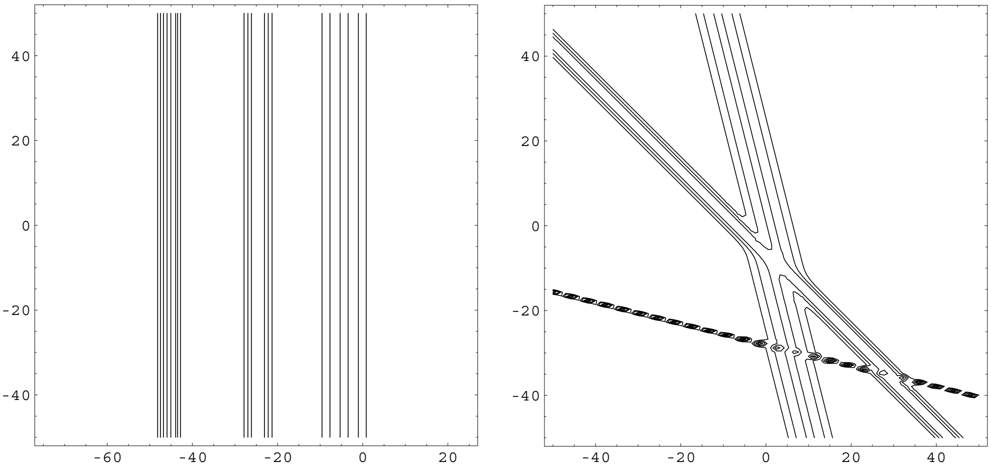

Figure 6. 3-soliton solutions of the KdV equation labelled by $\mathbf{n}^+ = (1, 2, 3)$ and $\mathbf{n}^- = (6, 5, 4)$. The left figure shows the 3-soliton solution in the x–y plane at $t = 25$ with $\theta_j = 0, \forall j$. The right one shows the solution in the x–t plane with different θ_j. The k_j are given by $(k_1, \ldots, k_6) = (-4/3, -1, -1/2, 1/2, 1, 4/3)$.

constraint $\partial u/\partial y = 0$ in the KP equation. Since line solitons are given by $\left[n_j^+, n_j^-\right]$-solitons for $j = 1, \ldots, N$, the constraint implies that all the solitons are parallel to the y-axis, i.e.

$$c_j = k_{n_j^+} + k_{n_j^-} = 0 \qquad \text{for all} \quad j = 1, \ldots, N.$$

Now from the ordering $k_1 < \cdots < k_{2N}$, we assume k_j to satisfy

$$k_1 < k_2 < \cdots < k_N < 0, \qquad k_{N+j} = -k_{N-j+1} \qquad \text{for} \quad j = 1, \ldots, N.$$

Then we take the set $(\mathbf{n}^+, \mathbf{n}^-)$ as

$$n_j^+ = j, \qquad n_j^- = 2N - j + 1 \qquad \text{for} \quad j = 1, \ldots, N,$$

which leads to $c_j = 0$ for all $j = 1, \ldots, N$. In terms of the matrix A_N, this corresponds to

$$A_N = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & * \\ 0 & 0 & \cdots & 0 & 0 & \cdots & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * & \cdots & 0 & 0 \end{pmatrix}.$$

This matrix indicates that any pair of line solitons are of P-type, that is, all the interactions are non-resonant. Each line soliton has the following form:

$$u(x, t) = 2k_j^2 \operatorname{sech}^2 \theta_j, \qquad \text{with} \quad \theta_j = -k_j x - k_j^3 t + \theta_j^0.$$

Thus each soliton has the velocity $\mathrm{d}x/\mathrm{d}t = -k_j^2$. We illustrate a 3-soliton solution of the KdV equation in figure 6.

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